Example: A : old assembly line , A′: new assembly line
B : defective , B′: not defective

Simple Events : $A\cap B$, $A'\cap B$, $A\cap B'$, $A'\cap B'$

	B	B′	
A	2	6	8
A′	1	9	10
	3	15	18

$P(B) = P$ (a randomly chosen product is defective) $= \frac{3}{18} = \frac{1}{6}$

$P(A) = P$ (old line) $= \frac{8}{18} = \frac{4}{9}$

$P(B|A) = P$ (defective | old line) $= \frac{2}{8} = \frac{1}{4} > \frac{1}{6} = P(B)$

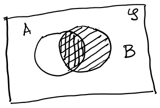
$P(A|B) = P$ (old line | defective) $= \frac{2}{3} > \frac{4}{9} = P(A)$

Now, $P(B|A) = \frac{2}{8} = \frac{2/18}{8/18} = \frac{P(A\cap B)}{P(A)}$

where $P(A\cap B) = P$ (a randomly chosen product is from old line and is defective)

$= \frac{2}{18}$

Similarly, $P(A|B) = \frac{2}{3} = \frac{2/18}{3/18} = \frac{P(A\cap B)}{P(B)}$



$S \longrightarrow S^n = B$

$P(A|B)$
$\propto P(A\cap B)$

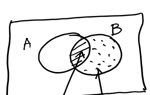
Just like $P(S) = 1$, we should have: $P(B|B) = 1$

$\Rightarrow P(A|B) = \frac{P(A\cap B)}{P(B)}$

Defn: For any two events A and B such that $P(B) > 0$, Conditional Prob. of A given B is: $P(A|B) = \frac{P(A\cap B)}{P(B)}$.

Properties:

① $B = \underbrace{(A\cap B)}_{\text{disjoint}} \cup \underbrace{(A^c\cap B)}_{\text{disjoint}}$



$\Rightarrow P(B) = P(A\cap B) + P(A^c\cap B)$

$\Rightarrow 1 = \frac{P(A\cap B)}{P(B)} + \frac{P(A^c\cap B)}{P(B)}$ (Assume: $P(B) > 0$)

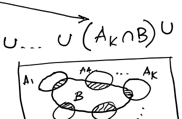
$\Rightarrow 1 = P(A|B) + P(A^c|B)$

$\Rightarrow \boxed{P(A^c|B) = 1 - P(A|B)}$

② $A_1, A_2, \ldots$ are disjoint
$P(B) > 0$

$(A_1\cup A_2\cup\ldots)\cap B = \underbrace{(A_1\cap B)}_{\text{disjoint}} \cup (A_2\cap B) \cup \ldots \cup (A_k\cap B) \cup \ldots$

$P[(A_1\cup A_2\cup\ldots)\cap B]$
$= P(A_1\cap B) + P(A_2\cap B) + \ldots$



$\Rightarrow \frac{P[(A_1\cup A_2\cup\ldots)\cap B]}{P(B)} = \frac{P(A_1\cap B)}{P(B)} + \frac{P(A_2\cap B)}{P(B)} + \ldots$

$\Rightarrow \boxed{P(A_1\cup A_2\cup\ldots | B) = P(A_1|B) + P(A_2|B) + \ldots}$

③ **Multiplication Rule:** $P(A|B) = \frac{P(A\cap B)}{P(B)}$

$\boxed{P(A\cap B) = P(A|B)\times P(B) = P(B|A)\times P(A)}$

(3) Multiplication Rule: $P(A|B) = \frac{P(A \cap B)}{P(B)}$

$$\boxed{P(A\cap B) = P(A|B) \times P(B) = P(B|A) \times P(A)}$$

Eg: 4 donors, Only one with type O+. Donors are tested in a random order.

$$P(\text{at least 3 are tested}) = P(\{1^{st} \text{ not } O^+\} \cap \{2^{nd} \text{ not } O^+\})$$

$$= P(1^{st} \text{ not } O^+) \times P(2^{nd} \text{ not } O^+ \mid 1^{st} \text{ not } O^+)$$

$$= \frac{3}{4} \times \frac{2}{3} = \frac{2}{4} = \frac{1}{2}$$

$$\boxed{P(A_1 \cap A_2 \cap A_3) = P(A_1) \times P(A_2|A_1) \times P(A_3 | A_1 \cap A_2)}$$ (check)

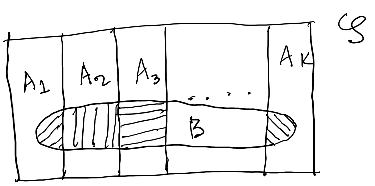
$$P(\text{exactly 3 are tested}) = P(\{1^{st} \text{ not } O^+\} \cap \{2^{nd} \text{ not } O^+\} \cap \{3rd\ O^+\})$$

$$= P(1^{st} \text{ not } O^+) \times P(2^{nd} \text{ not } O^+ \mid 1^{st} \text{ not } O^+)$$

$$\times P(3^{rd}\ O^+ \mid 1^{st} \,\&\, 2^{nd} \text{ not } O^+)$$

$$= \frac{3}{4} \times \frac{2}{3} \times \frac{1}{2} = \frac{1}{4}$$

- The Law of Total Prob:



$A_i \cap A_j = \phi \ \forall i \neq j \ (i, j = 1, 2, \dots, K)$

and $A_1 \cup A_2 \cup \dots \cup A_K = S$

i.e. $A_1, A_2, \dots, A_K$ are mutually exclusive and exhaustive (a partition of S)

$$B = (B \cap A_1) \cup (B \cap A_2) \cup \dots \cup (B \cap A_K)$$

$$\Rightarrow \boxed{\begin{aligned} P(B) &= P(B\cap A_1) + P(B \cap A_2) + \dots + P(B \cap A_K) \\ &= P(B|A_1) \times P(A_1) + P(B|A_2) \times P(A_2) \\ &\quad + \dots + P(B|A_K) \times P(A_K) \end{aligned}}$$

Eg. Blood donation Problem (contd.)

Donors : A, B, C, D
(A: O+)

P(at least 3 are tested) $= P$(A is tested at the 3rd place or 4th place)

$$= \frac{3! + 3!}{4!} = \frac{2 \times 3!}{4!} = \frac{1}{2}$$

P(exactly 3 are tested)

$$= \frac{3!}{4!} = \frac{1}{4}$$

$\{A, A^c\}$ is a partition of $\mathcal{S}$.

Law of Total Prob. $\Rightarrow P(B) = P(B \cap A) + P(B \cap A^c)$

$$= P(B) \times P(A|B) + P(B) \times P(A^c|B)$$

$$\Rightarrow \boxed{1 = P(A|B) + P(A^c|B)}$$

$$= P(A) \times P(B|A) + P(A^c) \times P(B|A^c)$$

Ex: Out of all VCR sales: 50% Brand 1, 30% Brand 2, 20% Brand 3. All 3 offer 1-year warranty. 25% of Brand 1, 20% of Brand 2 and 10% of Brand 3 VCR's require repair under warranty.

Let B_i = a random purchaser has Brand i VCR $(i = 1, 2, 3)$

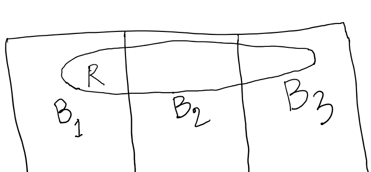
$\{B_1, B_2, B_3\}$ a partition of $\mathcal{S}$.

R = a random purchaser needed repair under warranty.

$P(B_1) = 0.5$, $P(B_2) = 0.3$, $P(B_3) = 0.2$.

$P(R|B_1) = 0.25$, $P(R|B_2) = 0.2$, $P(R|B_3) = 0.1$.

$\Rightarrow P(R^c|B_1) = 0.75$, $P(R^c|B_2) = 0.8$, $P(R^c|B_3) = 0.9$.



Q1: What's the Prob. that a randomly selected Purchaser has a Brand 1 VCR that needed repair?

$$P(B_1 \cap R) = P(B_1) \times P(R|B_1) = 0.125$$

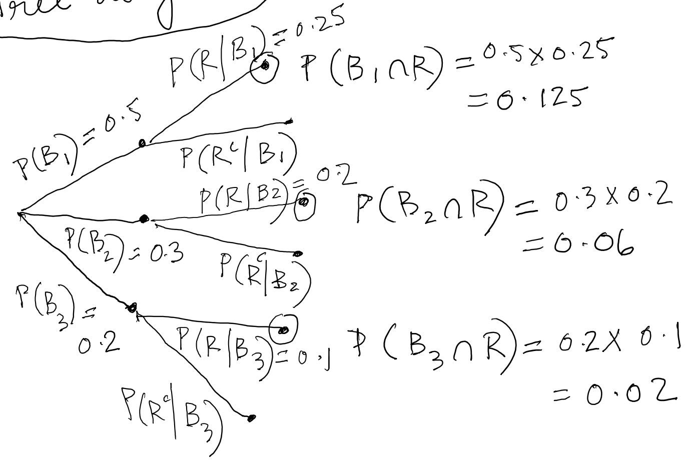
$$P(B_1 \cap R) = P(B_1) \times P(R|B_1) = 0.125$$

Q_2: What's the Prob. that a randomly selected purchaser bought a VCR that needed repair?

$$P(R) = P(R \cap B_1) + P(R \cap B_2) + P(R \cap B_3)$$

$$= 0.125 + 0.06 + 0.02 = 0.205$$

Tree diagram



$$P(R) = 0.205$$

Q3: If a customer returns with a VCR that needed repair, What's the Prob. that it's Brand 1/2/3?

Posterior Prob. (compare with prior prob. $P(B_1)$, $P(B_2)$ & $P(B_3)$)

$$P(B_1|R) = \frac{P(B_1 \cap R)}{P(R)} = \frac{P(R|B_1) \times P(B_1)}{\sum_{i=1}^{3} P(R|B_i) \times P(B_i)}$$

$$= \frac{0.125}{0.205} = 0.61$$

$$P(B_2|R) = \frac{0.06}{0.205} = 0.29$$

$$P(B_3|R) = \frac{0.02}{0.205} = 0.10$$

[Data = R]

Bayes Theorem:

Let $A_1, \ldots, A_K$ be mutually exclusive and exhaustive events. Then for any event B with $P(B) > 0$,

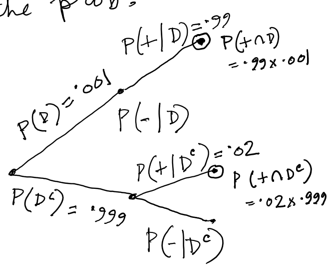
$$P(A_j \mid B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(B \mid A_j)\, P(A_j)}{\sum_{i=1}^{K} P(B \mid A_i)\, P(A_i)} \quad \forall j,\ 1 \le j \le K$$

w.l.o.g assume $P(A_j) > 0 \ \forall j$

Proof is by multiplication rule (num.) & Law of Total Prob. (denom.)

Example: (Incidence of a rare disease)

1 in 1000 has a rare disease. For those who have the disease, 99% have '+' test result and for those who don't have the disease, 2% have '+' result. If a randomly selected individual has '+' result, what's the prob. that the person has the disease?



D = disease

$+$ = '+' Test result

$-$ = '−' Test result

$$\boxed{\begin{array}{l} 99\% : \text{Sensitivity} \\ 98\% : \text{Specificity} \end{array}}$$

$$P(D \mid +) = \frac{P(+ \mid D)\, P(D)}{P(+ \mid D)\, P(D) + P(+ \mid D^c) \cdot P(D^c)}$$

$$= \frac{0.99 \times 0.001}{0.99 \times 0.001 + 0.02 \times 0.999}$$

$$= \frac{0.00099}{0.00099 + 0.01998} = 0.047$$

Independence:

Defn: Two events $\underbrace{A \text{ and } B}_{\text{assume } P(A), P(B) > 0}$ are indep. if

$$\boxed{P(A \mid B) = P(A)}$$

$$\iff \frac{P(A \cap B)}{P(B)} = P(A) \iff \boxed{P(A \cap B) = P(A) \cdot P(B)}$$

This defn. works if $P(A)$ or $P(B)$ or both $= 0$.

$$\iff P(B \mid A)\, P(A) = P(A)\, P(B)$$

$$\iff \boxed{P(B \mid A) = P(B)}$$

Three equivalent defn.s $\rightarrow$ Need to check only one

So if $P(A) = 0$, $P(A \cap B) = 0 = P(A) \cdot P(B)$

$\Rightarrow$ an impossible event is indep. of any event.

Eg. 1 A & B are mutually exclusive $[P(A), P(B) > 0]$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0}{P(B)} = 0 \neq P(A).$$

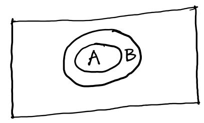
Eg. (1) A & B are mu…

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{0}{P(B)} = 0 \neq P(A).$$

$\Rightarrow$ A & B are dep.

<u>special case:</u> A and A^c are dep.

(2) $A \subset B$ such that $0 < P(A) < P(B) < 1$.



$A \cap B = A$

$\Rightarrow P(A\cap B) = P(A) > P(A).P(B)$

$\Rightarrow$ A & B are dep.

<u>Def:</u> $A_1, A_2, \dots, A_n$ are <u>mutually indep.</u> if for all k $(2 \le k \le n)$ and all subsets $\{A_{i_1}, A_{i_2}, \dots, A_{i_k}\}$ of $\{A_1, \dots, A_n\}$ of size k,

$$P(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_k}).$$

Here, $\{i_1, i_2, \dots, i_k\} \subset \{1, 2, \dots, n\}$

<u>Special case:</u> A, B, C are mutually indep. if

$$P(A\cap B) = P(A)P(B)$$
$$P(B\cap C) = P(B)\cdot P(C)$$
$$P(C\cap A) = P(C).P(A)$$
and $$P(A\cap B\cap C) = P(A).P(B).P(C).$$

<u>Example:</u> (3) Roll 2 fair dice. (36 equally likely outcomes)

A = 4 on Die 1

B = 3 on Die 2

C = Sum is 7

$A\cap B\cap C = \{(4,3)\}$

$A = \{(4,1), (4,2), \dots, (4,6)\}$

$B = \{(1,3), (2,3), \dots, (6,3)\}$

$A\cap B = \{(4,3)\}$, $B\cap C = \{(4,3)\}$, $A\cap C = \{(4,3)\}$

$C = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$

Pairwise indep.
$$\begin{cases} P(A\cap B) = \frac{1}{36} = \frac{1}{6}\times\frac{1}{6} = P(A).P(B) \\ P(B\cap C) = \frac{1}{36} = P(B).P(C) = \frac{1}{6}\times\frac{1}{6} \\ P(A\cap C) = \frac{1}{36} = P(A).P(C) = \frac{1}{6}\times\frac{1}{6} \end{cases}$$

$\left[P(A) = P(B) = P(C) = \frac{6}{36} = \frac{1}{6}\right]$

3-way dep. $\leftarrow$ $$P(A\cap B\cap C) = \frac{1}{36} \neq \underbrace{P(A)\times P(B)\times P(C)}_{\left(\frac{1}{6}\right)^3 = \frac{1}{216}}$$

NOT mutually indep.

④ A, B and C such that $P(A\cap B) \neq P(A).P(B)$ and $P(C) = 0$.

3-way indep. $\leftarrow$ $P(A\cap B\cap C) = 0 = P(A).P(B).P(C)$

But, $P(A\cap B) \neq P(A)P(B) \rightarrow$ 2-way dep. $\rightarrow$ NOT mutually indep.

⑤ Roll 2 fair dice.

A = 1, 2 or 3 on Die 1

B = 3, 4 or 5 on Die 1

C = Sum is 9.

Clearly, $P(A) = \frac{3}{6} = \frac{1}{2} = P(B)$ [By just looking at Die 1]

$C = \{(3,6), (4,5), (5,4), (6,3)\}$

$\therefore P(C) = 4/36 = 1/9$

$A\cap B$ = 3 on Die 1 $= \{(3,1), (3,2), \ldots, (3,6)\}$

$A\cap C = \{(3,6)\}$

$B\cap C = \{(3,6), (4,5), (5,4)\}$

$A\cap B\cap C = \{(3,6)\}$

3-way indep. $\leftarrow$ $P(A\cap B\cap C) = \frac{1}{36} = \frac{1}{2}\times\frac{1}{2}\times\frac{1}{9} = P(A)P(B)P(C)$

NOT mutually indep. $\leftarrow$ All 3 pairs are 2-way dep.

$$\begin{cases} P(A\cap B) = \frac{6}{36} = \frac{1}{6} \neq \frac{1}{2}\times\frac{1}{2} = P(A)P(B) \\ P(A\cap C) = \frac{1}{36} \neq \frac{1}{2}\times\frac{1}{9} = P(A)P(C) \\ P(B\cap C) = \frac{3}{36} = \frac{1}{12} \neq \frac{1}{2}\times\frac{1}{9} = P(B)P(C) \end{cases}$$

From examples 3, 4 and 5, we can say:

$$\boxed{\begin{array}{l} \text{3-way indep.} \not\Rightarrow \text{2-way indep.} \\ \text{2-way indep.} \not\Rightarrow \text{3-way indep.} \end{array}}$$

Lecture 10

Monday, June 26, 2017 1:00 PM

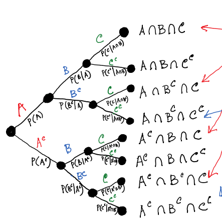
- If A and B are indep. then $P(A\cap B)=P(A)\cdot P(B)$.
  This works if you replace A or B by their complements.

$$P(A\cap B)=P(A)\cdot P(B)$$
$$\begin{aligned}P(A^c\cap B) &= P(B)-P(A\cap B)\\ &=P(B)-P(A)\cdot P(B)\\ &=P(B)[1-P(A)]\\ &=P(A^c)\cdot P(B)\end{aligned}$$

So, A and B^c, A^c and B and A^c and B^c are all (pairwise) indep.

Can be extended for multiple events. (>2)

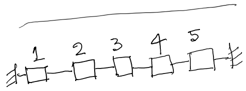
- Three-stage experiment:



$$P(C)=P(A\cap B\cap C)+P(A\cap B^c\cap C)+P(A^c\cap B\cap C)+P(A^c\cap B^c\cap C)$$

$$\begin{aligned}P(B^c\cap C^c) &= P(A\cap B^c\cap C^c)+P(A^c\cap B^c\cap C^c)\\ &= P(A)P(B^c|A)P(C^c|A\cap B^c)+P(A^c)P(B^c|A^c)P(C^c|A^c\cap B^c)\end{aligned}$$

- Serial Connection:



$$\begin{aligned}P(\text{System Fails}) &= 1-P(\text{System Succeeds})\\ &=1-P(S_1\cap S_2\cap\ldots\cap S_5)\\ &=1-P(S_1)P(S_2)\ldots P(S_5)\\ &=1-(0.9)^5\end{aligned}$$

$[S_i$'s are indep. $P(S_i)=0.9]$

$$P(\text{System Fails})=P(F_1\cup F_2\cup\ldots\cup F_5)$$
$$=\underbrace{\sum_{i=1}^{5}P(F_i)}_{\binom{5}{1}\text{ terms}}-\underbrace{\sum_{i<j}\sum P(F_i\cap F_j)}_{\binom{5}{2}\text{ terms}}+\ldots+\underbrace{P(F_1\cap\ldots\cap F_5)}_{\binom{5}{5}\text{ terms}}=5\times P(F_1)-\binom{5}{2}\times P(F_1)\times P(F_2)+\ldots+P(F_1)\times\ldots\times P(F_5)$$
$$=5\times(0.1)-\binom{5}{2}\times(0.1)^2+\ldots+(0.1)^5=1-(0.9)^5$$ (Check!)

$$\begin{aligned}P(\text{exactly 1 component fails}) &= P(F_1\cap S_2\cap S_3\cap S_4\cap S_5)+P(S_1\cap F_2\cap S_3\cap S_4\cap S_5)\\ &\quad+\ldots+P(S_1\cap S_2\cap\ldots\cap S_4\cap F_5)\\ &=P(F_1)P(S_2)\ldots P(S_5)+\ldots+P(S_1)\ldots P(F_5)\\ &=(0.1)(0.9)^4+\ldots+(0.1)(0.9)^4\\ &=5(0.1)(0.9)^4\end{aligned}$$

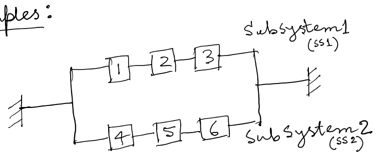
Similarly, $P(\text{exactly 2 components fail})=\binom{5}{2}(0.1)^2(0.9)^3$

In general, $P(\text{exactly } j \text{ components fail})=\binom{5}{j}(0.1)^j(0.9)^{5-j}$ $[j=0,1,\ldots,5]$

$$\begin{aligned}P(\text{System fails}) &= P(\{\text{exactly 1 component fails}\}\\ &\quad\cup\{\text{exactly 2 components fail}\}\\ &\quad\cup\ldots\cup\{\text{exactly 5 components fail}\})\\ &=\sum_{j=1}^{5}\binom{5}{j}(0.1)^j(0.9)^{5-j}=1-(0.9)^5\end{aligned}$$ (Check!)

- Examples:

1.



System Works if at least $\{1,2\&3\}$ or $\{4,5\&6\}$ work.

A_i: lifetime of component $i>T_0$ hours $(i=1,\ldots,6)$

$P(A_i)=0.9\quad\forall i,\ 1\le i\le 6$

All components are indep.

$=?$

$P(A_i) = 0.9 \quad \forall i, 1 \le i \le 6$

All components are indep.

$$P(\text{System lifetime} > T_0 \text{ hours}) = P(\{SS1 \text{ lifetime} > T_0\} \cup \{SS2 \text{ lifetime} > T_0\})$$

$$= P(\{A_1 \cap A_2 \cap A_3\} \cup \{A_4 \cap A_5 \cap A_6\})$$

$$= P(A_1 \cap A_2 \cap A_3) + P(A_4 \cap A_5 \cap A_6) - P(A_1 \cap A_2 \cap \ldots \cap A_6)$$

$$= (0.9)^3 + (0.9)^3 - (0.9)^6 = 0.927$$

$$P(\text{System lifetime} > T_0) = 1 - P(\text{System lifetime} \le T_0)$$

$$= 1 - P(\{SS1 \text{ lifetime} \le T_0\} \cap \{SS2 \text{ lifetime} \le T_0\})$$

$$= 1 - P(SS1 \text{ lifetime} \le T_0)\, P(SS2 \text{ lifetime} \le T_0)$$

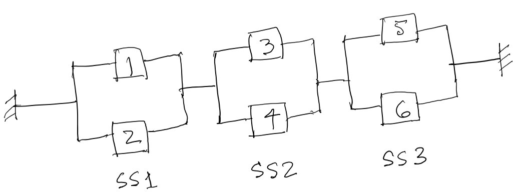
$$= 1 - P[A_1^c \cup A_2^c \cup A_3^c]\, P[A_4^c \cup A_5^c \cup A_6^c]$$

De Morgan's Law

$$= 1 - [1 - P(SS1 \text{ lifetime} > T_0)][1 - P(SS2 \text{ lifetime} > T_0)]$$

$$= 1 - [1 - P(A_1 \cap A_2 \cap A_3)] \times [1 - P(A_4 \cap A_5 \cap A_6)]$$

$$= 1 - (1 - 0.9^3)^2 = 0.927$$

2. 

$P(A_i) = 0.9 \quad \forall i, 1 \le i \le 6$

$$P(\text{System lifetime} > T_0) = P(\{SS1 \text{ lifetime} > T_0\} \cap \{SS2 \text{ lifetime} > T_0\} \cap \{SS3 \text{ lifetime} > T_0\})$$

$$= P(SS1 \text{ lifetime} > T_0)\, P(SS2 \text{ lifetime} > T_0)\, P(SS3 \text{ lifetime} > T_0)$$

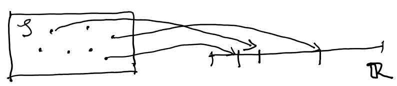
$$= [0.9 + 0.9 - 0.9^2]^3$$

$$= 0.970$$

Lecture 11

Tuesday, June 27, 2017 1:23 PM

Chapter 3: Discrete Random Variables (RV's)

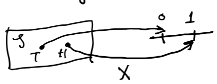


A RV is any rule that associates each outcome in $\mathcal{S}$ with a real number.

Eg: Flip of a coin $\mathcal{S} = \{H, T\}$ $\quad X = \begin{cases} 1 & \text{if } H \\ 0 & \text{if } T \end{cases}$

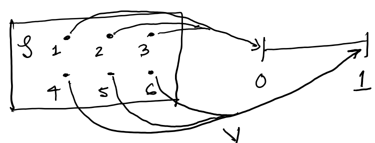
Repeat n times $\rightarrow X_1 + \dots + X_n = \#\,H\text{'s}$.

$Y = \begin{cases} 1 & \text{if } H \\ -1 & \text{if } T \end{cases}$ $\quad Y_1 + \dots + Y_n \approx 0$ as $n \to \infty$



2) Roll a die, $X = \#$ facing up

$X = \begin{cases} 1 & \text{if } ⚀ \\ 2 & \text{if } ⚁ \\ 3 & \text{if } ⚂ \\ \vdots \\ 6 & \text{if } ⚅ \end{cases}$ $\quad Y = \begin{cases} 1 & \text{if } X \geq 4 \\ 0 & \text{if } X \leq 3 \end{cases}$



__Def^n:__ A RV that only takes the values 0 and 1, is called a Bernoulli RV. (irrespective of the expt.) e.g. X in eg 1 and Y in eg. 2.

Discrete RV: # of possible values is finite / countably infinite.

Continuous RV: RV that can take any value from an interval or a collection of intervals. e.g. Temperature on a given day at a random time.

Prob. of each outcome in $\mathcal{S} \Rightarrow$ Prob. of each value that X takes.

The set of possible values of X: Domain of X (Not^n: D_X)

Prob. dist^n: Prob. for each value in D_X.